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# Upper and lower critical dimension of the three-state Potts spin glass

G Schreider† and J D Reger
Johannes Gutenberg Universität Mainz, Institut für Physik, D-55099 Mainz, Germany

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**Abstract.** We used a high-temperature star-graph expansion to compute the susceptibility of a $q$-state Potts glass with a bimodal distribution of the random bonds. The series are up to order O(20) in $K = \beta J$. In three dimensions our result shows excellent consistency with existing Monte Carlo data. The results for the three-state Potts glass in three to ten dimensions are analysed using dlog Padé approximants and the Adler–Moshe–Privman (AMP) method in combination with variable transformations. The analysis supports $d = 3$ as lower critical dimension with an exponential singularity at $T_c = 0$, as concluded earlier using Monte Carlo data. According to our analysis the upper critical dimension is $d = 8$, which is in contrast to earlier papers.

## 1. Introduction

Spin systems with competing interactions which are randomly distributed often show a glassy state at sufficiently low temperatures. The related concepts are used more and more to describe so-called orientational glasses [1] such as mixed crystals of K(Br,CN), mixed ortho- and para-hydrogen or nitrogen diluted with argon. In these systems only one species has an orientational degree of freedom. If one confines the number of possible orientations to a finite number, it is quite intuitive to use spins for modelling this situation. The $q$-state Potts glass can serve as a generic model for such systems. Symmetries play a very important role in critical phenomena. In the Ising case we have a spin-inversion symmetry, which means that flipping all spins of a systems without an external magnetic field into their opposite directions yields the same energy. However, this is not true for the Potts glass for $q \geqslant 3$, where $q$ is the number of possible directions the spin can point to, simply because there is no unique 'opposite' direction of an orientation. There are a lot of systems which do not have spin-inversion symmetry so these systems could be modelled by the Potts glass. Besides, it is a natural extension to the Ising glass, which has been studied very extensively [2]. There are a lot of essential differences to the Ising case which make this model far more complex and rich in behaviour. First of all, as already mentioned, there is no spin-inversion symmetry. The frustration in the model depends on the number of states $q$ a spin can have—it disappears in the limit $q \to \infty$. For $q > 2$ the antiferromagnetic system has no unique ground state; this results in a non-vanishing ground-state entropy of the glass [3]. Furthermore, we have a rich behaviour through combinatorial effects which are trivial in the Ising case. In the Landau expansion of the Hamiltonian the fourth-order term changes sign at $q \approx 2.8$ [1,4]; this is responsible for the difference between the Ising

† Present address: SAP AG, Neurottstr. 16, D-69190 Walldorf, Germany.

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case and the general Potts case in the mean-field (MF) treatment. For large enough $q$ in MF a *first-order phase transition* occurs [4].

We studied the $q$-state Potts glass defined by the Hamiltonian

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{i,j}\, \delta_{s_i,s_j} . \tag{1.1}$$

The sum runs over all nearest-neighbour pairs of spins on the lattice. The spins can be in $q$ different states. The spin–spin interaction energies $J_{i,j}$ are quenched random variables. The distribution is chosen to be bimodal:

$$P(J_{i,j}) = \tfrac{1}{2}(\delta(J_{i,j} - J) + \delta(J_{i,j} + J)) . \tag{1.2}$$

We studied a static magnetic response function, the Edwards–Anderson (EA) susceptibility, defined as [1]

$$\frac{1}{\beta}\chi = \sum_{\langle i,j \rangle} \left[ \left\langle \frac{1}{(q-1)} (q\delta_{s_i s_j} - 1) \right\rangle^2 \right]_{\rm av} . \tag{1.3}$$

The angular brackets refer to thermal averaging, while square brackets refer to an average over the disorder. We computed the inverse susceptibility by means of a star-graph expansion to order O(10) in $K^2$ for arbitrary $q$ and for arbitrary dimension $d$ of a hypercubic lattice. The method is described in detail in a previous paper [5], where the series are published in the appendix. In this paper we will analyse the results for the three-state Potts glass in order to estimate the upper and lower critical dimension of the model. We will then discuss our findings in the light of other results.

## 2. Results

For a first impression of the data the susceptibility series in three to ten dimensions are plotted as functions of the temperature (figure 1). The series in three and four dimensions bend over, which is an artefact of working with a truncated series and indicates the negative sign of the last term in the series.

According to the changing signs of the terms in the series below five dimensions, one finds this bending for series of order 8, 10 and so on. This irregular change of sign in the terms of the series is the reason for the failure of analysis methods, based on the ratio method.

The result in three dimensions can be compared with known Monte Carlo (MC) data [6]. In figure 2 there is *no* rescaling done; just the raw data were plotted. We find a very good agreement if some higher Padé approximants, $\left[\frac{3}{4}\right]$, $\left[\frac{5}{4}\right]$ or $\left[\frac{6}{3}\right]$, are used for comparison with the raw MC data. Additionally to the Padé approximants we plotted the series in nine and ten terms.

In order to find the right singularity approximated by this finite expansion, we used our knowledge of how this function should look. It is generally assumed that such observables can be described by scaling functions $F(p)$ which show a simple power-law behaviour at the critical temperature.

$$F(p) \sim A(p_c - p)^{-\gamma} \qquad p \to p_c . \tag{2.1}$$

If one wants to look for power-law singularities in the data then the Padé approximant method is the right choice. There are problems applying this method to series where one

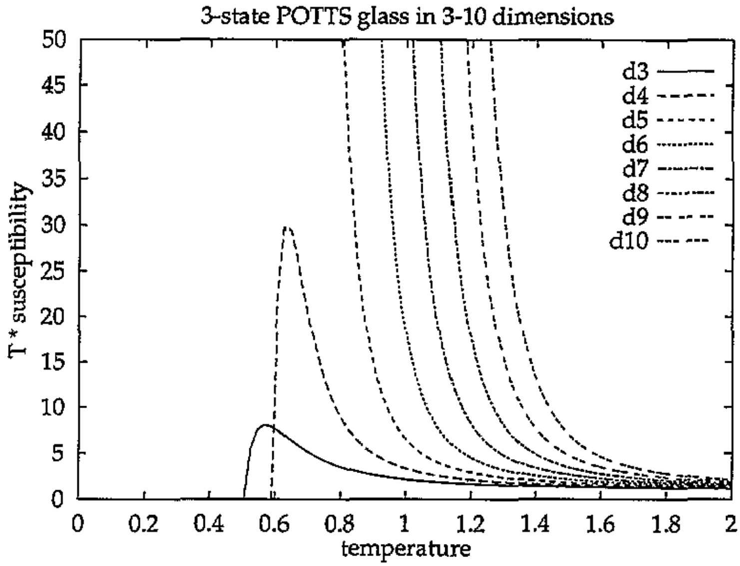


**Figure 1.** The series with ten terms from the high-temperature expansion of (1.3) in different dimensions (3D–10D) on hypercubic lattices plotted as a function of the temperature.

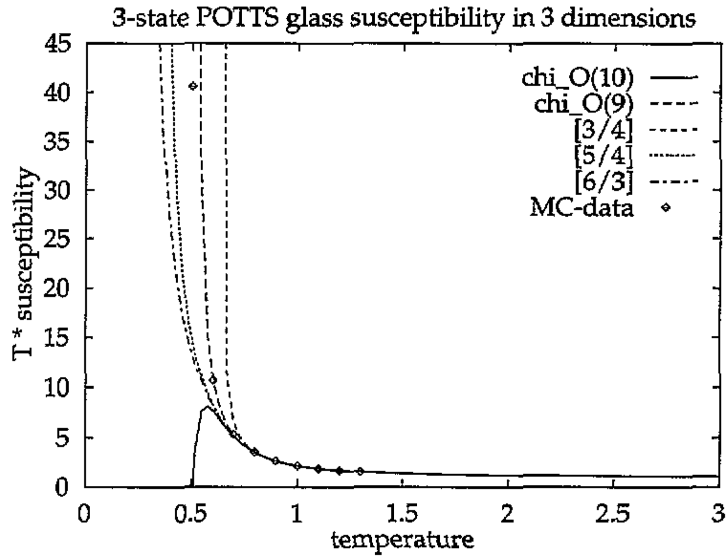


**Figure 2.** Monte Carlo data from [6] in comparison to the series and some Padé approximants.

has confluent singularities but it works well if they are small. A Padé approximant $[L/M]$ of a series is a rational function [7]

$$[L/M] = \frac{r_0 + r_1 p + r_2 p^2 + \cdots + r_L p^L}{1 + s_1 p + s_2 p^2 + \cdots + s_M p^M} \tag{2.2}$$

where the first $N = (L + M + 1)$ coefficients of the Taylor expansion of $[L/M]$ are determined in such a way that they are equal to the coefficients of the series. In this way one can compute a table of rational functions with $(N-1)^2$ entries. The singularities of the function $F(p)$ are represented by the poles of the rational functions. If the assumed function $F(p)$ is of the form (2.1) then the logarithmic derivative is

$$\frac{\mathrm{d}}{\mathrm{d}p} \ln F(p) \sim \frac{-\gamma}{(p_c - p)} . \tag{2.3}$$

Computing the Padé approximants for this function should result in a pole at $p_c$ and a corresponding residue of $-\gamma$. If the numerator has a zero at the position of a pole, i.e. a zero of the denominator, this is called a defective approximant and it is discarded in the analysis. A rational function can have many poles, but only the one next to zero is assumed to be the physical one. If one sees more or less the same pole and residue for different approximants then the analysis can be trusted. The physical singularity, i.e. the transition temperature, is found and the corresponding residue is the critical exponent. This works well for all series in ten to six dimensions. We can get estimates of the critical temperature and the critical exponent as well within an accuracy of 2–5% with our data.

We can refine these results further. To this end it is assumed that the function does not show the simple power law (2.1) but a power law with higher-order correction terms as they are assumed from renormalization-group calculations

$$F(p) \sim A(p_c - p)^{-\gamma}\left[1 + A_1(p_c - p)^{\Delta_1} + A_2(p_c - p) + \cdots\right] \qquad p \to p_c . \tag{2.4}$$

In order to analyse functions with this scaling form one uses a variable transformation technique known as the AMP method [7, 8]. The use of modern graphical analysis tools [9] enables one to analyse the given series in a very accurate way. For a given set of

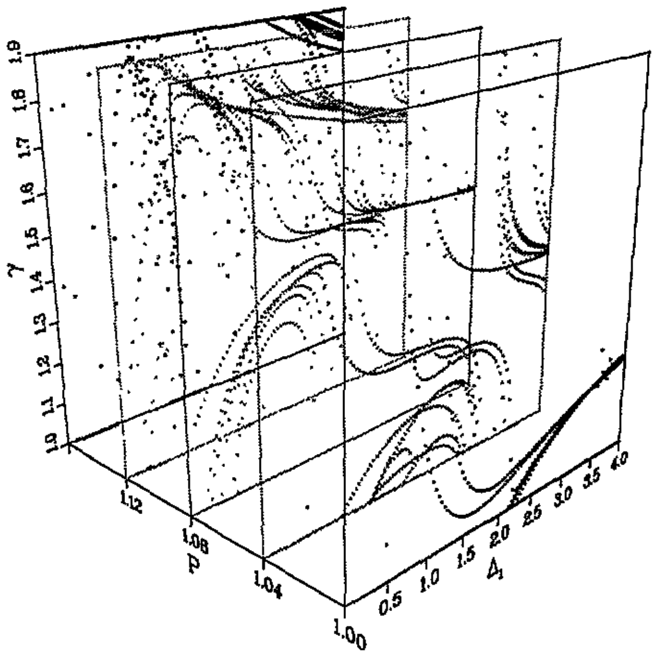


**Figure 3.** AMP analysis of the susceptibility of the three-state Potts glass in $d = 6$ on a hypercubic lattice. Parameters as in (2.4).

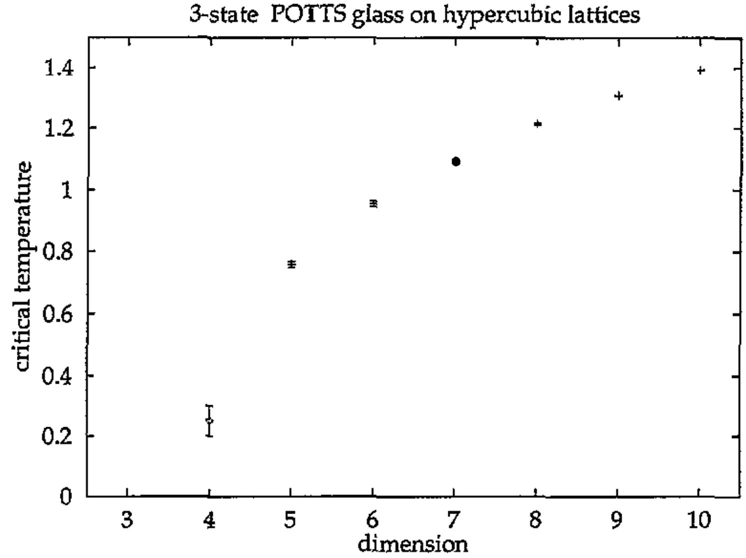


**Figure 4.** The critical temperature from the AMP analysis versus lattice dimension, added from [6] the MC result in $d = 4$.

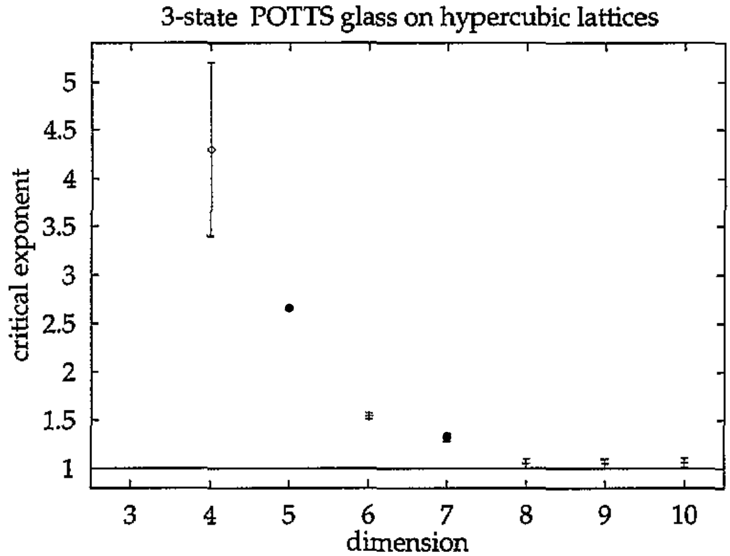


**Figure 5.** The critical exponent $\gamma$ from the AMP analysis versus lattice dimension, added from [6] the MC result in $d = 4$, $\gamma = (2 - \eta)\nu$ is used to compute $\gamma$ from the MC data.

parameters ($p_c$ and $\Delta_1$) of the variable transformation one calculates new series and their Padé approximants, as well as their poles and residues as stated above. It is known that the 'right' choice of the parameters, i.e. the ones that fit the series best to the estimated behaviour, appears as a converging of the curves in the plot [9]. An example of such behaviour is shown in figure 3, where the data for dimension $d = 6$ are shown. For a set of five values of the assumed critical temperature $p = (1.00–1.16)$ and for several values of

$\Delta_l$ the residues $\gamma$ are calculated. Each point in the figure represents a residue. Each plane in the figure belongs to one assumed critical temperature. The converging of the curves can be seen in the centre plane at the critical temperature $p = 1.08$. A two-dimensional plot of this plane enables us to determine the value of $\gamma$ to be $\gamma = 1.54 \pm 0.03$. This is well above the mean-field value of 1.0 which is clearly outside the error bars of this analysis.

This analysis was carried out for dimensions 5–10. The values of the critical temperature and the critical exponent are plotted as function of the lattice dimension in figures 4 and 5.

One sees a smooth behaviour of the values as functions of the dimension. The critical temperature decreases from 1.4 in ten dimensions to 0.8 in five dimensions. This can be understood as follows. If we assume that there is one constant energy scale for the critical coupling constant, then reducing the coordinate number of the lattice, $z$, has to be compensated by reducing the critical temperature, $T_c$, in order to keep the coupling constant:

$$\frac{zJ}{k_B T_c} = K_c . \tag{2.5}$$

Our result for the critical temperatures in five to eight dimensions agrees within the error bars with the results of Singh [10], but the exponents he found are different. It is strange that the exponents he got below $d = 6$ are smaller than unity, which indicates to us that there is a problem with his series. Additionally, his series in $d = 3$ differs from the MC data below 0.7 [6], although his series has one term more than ours, whereas we do find excellent agreement of our series with the MC data down to 0.6.

The behaviour of the critical exponent $\gamma$ is even more interesting. Above eight dimensions we do have mean-field behaviour, but below we see a value of $\gamma$ which clearly exceeds 1.0 and is rising as the dimension is further reduced. If we assume that mean-field behaviour is defined such that all critical exponents have mean-field values simultaneously, this means that the upper critical dimension of the three-state Potts glass is 8. This is a new result, which is in contrast to earlier papers which assume that the upper critical dimension of the Potts glass is 6. This latter assumption is a sole extension of the Ising-glass result, which was obtained by using the hyperscaling relation

$$2\beta + \gamma = d_c \nu \tag{2.6}$$

and inserting the mean-field values of the exponents $\beta = \gamma = 1$, $\nu = \frac{1}{2}$ [11]. This argument led to the right value of the upper critical dimension of the Ising spin glass $d_c = 6$ and it was *assumed* that the same is true for the Potts glass. This is the reason why in field-theoretical studies $(6 - \epsilon)$-expansions are done. However, since the mean-field results for the Potts glass already differ in essential points from the Ising glass [4], we doubt that the above-mentioned assumption is justified. In the light of our result it would be better to do an $(8 - \epsilon)$-expansion. There are also some field-theoretical results which give some indication that 6 might not be the true upper critical dimension for the Potts glass [12, 13]. They were interpreted as a fluctuation-driven phase transition of first order for values of $q < 3.77$ below eight dimensions.

So far we have analysed the data in higher dimensions, but what about 'physically relevant' dimensions? As we can see from the dlog Padé analysis, the method breaks down in four and three dimensions. There are mainly two possible explanations for that: firstly, the data are not good enough to see the asymptotic behaviour; secondly, there is no singularity that can be detected by this method. The first possibility is difficult to understand, because in higher dimensions the data did show good results, therefore the second possibility will serve as a basis for our further investigations. What kind of 'other' singularities might be present in the susceptibility which cannot be detected by the AMP method? We think that the strongest singularities might be exponential singularities. There are some other indications that this kind of singularity might indeed be the one we face in the three-dimensional case. If we look at the dependence of the critical exponent on the lattice dimension we see a steep rising of the values with falling dimension. If an exponential singularity is present in three dimensions, this would be indicated by a value of infinity in figure 5, because an exponential singularity is stronger than any power-law singularity. This steep rise may be a first indication of a behaviour of this kind. Another hint comes from a theoretical point of view. There is a simple scaling theory for spin glasses with symmetric bond distributions [14] which predicts a singularity of the correlation length like

$$\xi_{\mathrm{SG}} \sim \exp(cK^{\sigma}) \tag{2.7}$$

at the lower critical dimension. Using the relation between the correlation length and the susceptibility, which holds for any dimension

$$\chi_{\mathrm{SG}} \sim \xi_{\mathrm{SG}}^{2-\eta} \tag{2.8}$$

a similar behaviour should be seen in the spin-glass susceptibility.

If we could transform this kind of singularity into an ordinary power-law singularity we could see it with our available tools. Let us have a look at the natural variable emerging in the computation of the susceptibility. In the case of the Ising model this is the hyperbolic tangent. In the $q$-state Potts model it is

$$\int P(K')\left(\frac{\exp(K')-1}{\exp(K')+(q-1)}\right)\mathrm{d}K' . \tag{2.9}$$

This reduces to $\tanh(K^2)$ in the $q = 2$ case with a bimodal distribution. The tanh is an odd-symmetric function. In the general case $q \neq 2$ there is no symmetry at all. Expanding $\tanh(K^2)$ for $T \to \infty \Leftrightarrow K \to 0$ gives

$$\tanh(K^2) \sim 1 - 2\exp\left(-2K^2\right). \tag{2.10}$$

We want to do a biased analysis of our data for the presence of such a singularity. Using the new variable $y := \tanh(K^2)$ in our transformed series we can do an AMP analysis. The result is shown in figure 6, which shows good convergence of the curves at a critical value of $y_c = 1$.

With the help of (2.10) this value transforms into

$$1 = y_c = 1 - \exp\left(-2K_c^2\right) \tag{2.11}$$

which yields a critical temperature of *exactly* $T_c = 0$. This is a strong indication that $d = 3$ is indeed the lower critical dimension and that we do have a exponential singularity of the form

$$\chi_{\mathrm{SG}} \sim \exp\left(cK^2\right). \tag{2.12}$$

We also did biased analyses for $\exp(K)$ and $\exp(K^3)$ but could not find a clear cut singularity. Thus our data confirm the result found independently by MC simulation [6] that there is a exponential singularity at $T_c = 0$ with a non-trivial exponent $\sigma = 2$. This behaviour of the three-state Potts glass with bimodal distribution is consistent with the scaling predictions of McMillan [14] for a spin glass at its lower critical dimension.

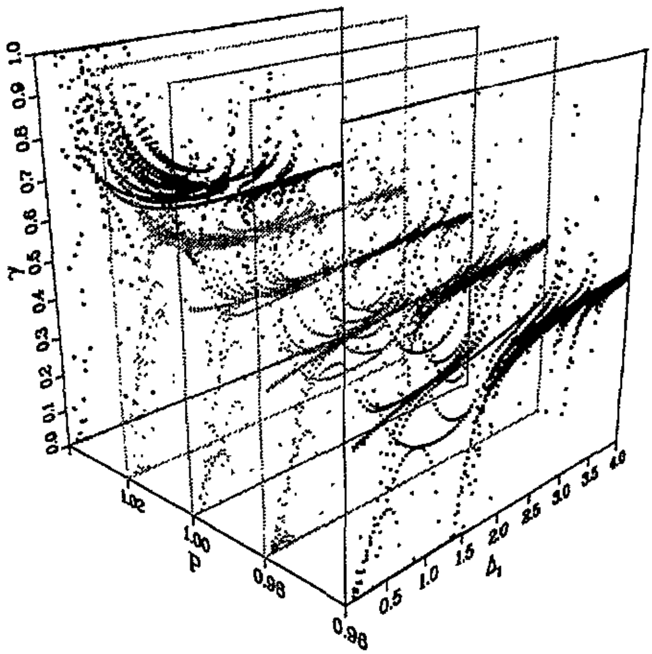


**Figure 6.** AMP analysis of the in $d = 3$ on a cubic lattice. Parameters as in (2.4).

## 3. Outlook

It is planned to analyse the series for other numbers of spin orientations $q$. By this means it may be possible to obtain a map of the critical exponent $\gamma$ in the $(q, d)$-plane. This will help us to understand the dependence of the type of phase transition on typical model parameters, as it is known that the $q$-state Potts ferromagnets show different phase transitions depending on the location of the model in the $(q, d)$-plane [15] and it is widely believed that this also applies to Potts glasses.

## Acknowledgments

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